

ON ABSOLUTE CONTINUITY OF THE SPECTRUM OF PERIODIC SCHRÖDINGER OPERATORS

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ABSTRACT. In this paper we find a new condition on a real periodic potential for which the self-adjoint Schrödinger operator may be defined by a quadratic form and the spectrum of the operator is purely absolutely continuous. This is based on resolvent estimates and spectral projection estimates in weighted L^2 spaces on the torus, and an oscillatory integral theorem.

1. INTRODUCTION

The behavior of a non-relativistic quantum particle is described by the wave function $\Psi(t, x)$ which is governed by the Schrödinger equation

$$i\partial_t \Psi(t, x) = H\Psi(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3,$$

where $H = -\Delta + V(x)$ is the Schrödinger operator and V is the potential which is a real function. In view of spectral theory, the solution can be given by $\Psi(t, x) = e^{itH}\Psi(0, x)$ if H is self-adjoint. In this regard, spectral properties of self-adjoint Schrödinger operators have been extensively studied since the beginning of quantum mechanics.

In this paper we are mainly concerned with the problem of finding conditions on a real periodic potential V for which the spectrum of H is purely absolutely continuous. More generally, we will consider the following differential operator:

$$DAD^T + V(x), \quad x \in \mathbb{R}^3, \quad (1.1)$$

where $D = -i\nabla$ and $A = (a_{jk})$ is a symmetric, positive-definite 3×3 matrix with real constant entries. Here, we are using DAD^T to denote $\sum_{j,k=1}^3 D_j a_{jk} D_k$, and V is a real periodic function which means that $V(x + e_j) = V(x)$ for some basis $\{e_j\}_{j=1}^3$ of \mathbb{R}^3 . Note that $DAD^T = -\Delta$ particularly when $A = I$ is the identity matrix, and we may choose $e_1 = 2\pi(1, 0, 0)$, $e_2 = 2\pi(0, 1, 0)$ and $e_3 = 2\pi(0, 0, 1)$ by a change of variables. Namely, V is assumed to be periodic with respect to the lattice $(2\pi\mathbb{Z})^3$.

Let $\Omega = [0, 2\pi)^3$ be a cell of the lattice, and for $N \geq 0$ let $V_N(x) = V(x)$ if $|V(x)| > N$ and $V_N(x) = 0$ if $|V(x)| \leq N$. Let \mathcal{F} be a function class equipped with the norm

$$\|f\|_{\mathcal{F}} := \sup_{z \in \mathbb{R}^3, r > 0} \left(\int_{Q(z, r)} |f(x)| dx \right)^{-1} \int_{Q(z, r)} \int_{Q(z, r)} \frac{|f(x)f(y)|}{|x - y|} dx dy,$$

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where $Q(z, r)$ denotes dyadic cubes in \mathbb{R}^3 centered at z with side length r . Now we will consider potentials $V \in L^1_{\text{loc}}(\mathbb{R}^3)$ such that for a sufficiently small $\varepsilon > 0$

$$\lim_{N \rightarrow \infty} \|V_N \chi_\Omega\|_{\mathcal{F}} < \varepsilon, \quad (1.2)$$

where χ_Ω is the characteristic function of the cell Ω . Note that the condition (1.2) is equivalent to

$$\lim_{N \rightarrow \infty} \sup_{z \in \Omega, 0 < r < 4\pi} \left(\int_{Q(z, r)} |V_N(x)| dx \right)^{-1} \int_{Q(z, r)} \int_{Q(z, r)} \frac{|V_N(x)V_N(y)|}{|x - y|} dx dy < \varepsilon,$$

because V_N is also periodic with respect to $(2\pi\mathbb{Z})^3$, and x, y is taking only in Ω .

Let us now consider the following quadratic form to define the self-adjoint operator $DAD^T + V$:

$$q[f, g] = \int_{\mathbb{R}^3} \langle (Df)A, Dg \rangle + \langle Vf, g \rangle dx, \quad f, g \in C_0^1(\mathbb{R}^3), \quad (1.3)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in \mathbb{C} or in \mathbb{C}^3 . Then we have

Theorem 1.1. *Let $V \in L^1_{\text{loc}}(\mathbb{R}^3)$ be a real periodic function with respect to the lattice $(2\pi\mathbb{Z})^3$. If V satisfies (1.2), then there exists a unique self-adjoint operator denoted by $DAD^T + V$ such that*

$$q[f, g] = \int_{\mathbb{R}^3} \langle (DAD^T + V)f, g \rangle dx$$

for $f \in \text{Dom}[DAD^T + V]$ and $g \in H^1(\mathbb{R}^3)$. Here, the domain of $DAD^T + V$ is

$$\text{Dom}[DAD^T + V] = \{f \in H^1(\mathbb{R}^3) : (DAD^T + V)f \in L^2(\mathbb{R}^3)\}.$$

Let $\mathcal{M}^p(\mathbb{R}^n)$ denote the Morrey class of functions f on \mathbb{R}^n which is defined by the norm

$$\|f\|_{\mathcal{M}^p(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n, r > 0} |Q|^{2/n} \left(\frac{1}{|Q|} \int_{Q(x, r)} |f(y)|^p dy \right)^{1/p} < \infty, \quad 1 \leq p \leq n/2.$$

Note that $\mathcal{M}^{n/2}(\mathbb{R}^n) = L^{n/2}(\mathbb{R}^n)$ and $1/|x|^2 \in L^{n/2, \infty}(\mathbb{R}^n) \subset \mathcal{M}^p(\mathbb{R}^n)$ if $p < n/2$. In [27], the self-adjointness of (1.1) was shown for $V \in \mathcal{M}^p(\mathbb{R}^n)$ with $p > (n-1)/2$, $n \geq 3$, under the smallness assumption

$$\limsup_{r \rightarrow 0} \|V\|_{\mathcal{M}^p(\mathbb{R}^n)} < \varepsilon \quad (1.4)$$

which implies (see Lemma 2.7 in [27])

$$\lim_{N \rightarrow \infty} \|V_N \chi_\Omega\|_{\mathcal{M}^p(\mathbb{R}^n)} < \varepsilon. \quad (1.5)$$

Remark 1.2. Here we shall explain that (1.5) with $n = 3$ implies (1.2). Thus Theorem 1.1 improves the one of [27] in three dimensions. In fact, our motivation for the function class \mathcal{F} stemmed from the characterization of the weighted L^2 inequalities

$$\|I_1 f\|_{L^2(w)} \leq C_w \|f\|_{L^2}, \quad (1.6)$$

where I_α denotes the fractional integral operator of order $0 < \alpha < n$:

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy. \quad (1.7)$$

The class \mathcal{F} and the characterization of (1.6) has been recently used in various problems concerning Schrödinger operators and equations ([1, 22, 23, 24]). As is well known from [8], (1.6) holds for $w \in \mathcal{M}^p(\mathbb{R}^n)$ with $C_w \sim \|w\|_{\mathcal{M}^p(\mathbb{R}^n)}^{1/2}$ if $p > 1$. But here, we note that the least constant C_w which allows (1.6) with $n = 3$ may be taken to be a constant multiple of $\|w\|_{\mathcal{F}}^{1/2}$ (see Lemma 2.1 for more details). It is therefore clear that (1.5) with $n = 3$ implies (1.2).

Now we turn to the absolute continuity. We first need to set up more notation. A weight $w : \mathbb{R}^3 \rightarrow [0, \infty]$ is said to be of Muckenhoupt $A_2(\mathbb{R}^3)$ class (cf. [10]) if there is a constant C_{A_2} such that

$$\sup_{Q \text{ cubes in } \mathbb{R}^3} \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1} dx \right) < C_{A_2}.$$

Note that $w \in A_2 \Leftrightarrow w^{-1} \in A_2$. Given $v \in \mathbb{R}^3$, one can write for $x \in \mathbb{R}^3$, $x = \lambda v + \tilde{x}$, where $\lambda \in \mathbb{R}$ and \tilde{x} is in some hyperplane \mathcal{P} whose normal vector is v . We shall denote by $w \in A_p(v)$ to mean that w is in the A_2 class in one-dimensional direction of the vector v if the function $w_{\tilde{x}}(\lambda) := w(x)$ is in $A_2(\mathbb{R})$ with C_{A_2} uniformly in almost every $\tilde{x} \in \mathcal{P}$. By translation and rotation, this notion can be reduced to the case where $v = (0, 0, 1) \in \mathbb{R}^3$ and $\mathcal{P} = \mathbb{R}^2$. In this case, $w \in A_2(v)$ means that $w(x_1, x_2, \cdot) \in A_2(\mathbb{R})$ with respect to the variable x_3 uniformly for $\tilde{x} = (x_1, x_2) \in \mathbb{R}^2$. Also, $w \in A_2(v)$ is trivially satisfied if w is in a more restrictive $A_2(\mathbb{R}^3)$ class defined over rectangles instead of cubes (see Lemma 2.2 in [17]).

This one-dimensional A_2 condition was already appeared in the study of unique continuation problems (cf. [6, 23]), and will be needed here for our resolvent estimates in Proposition 3.2 which is a key ingredient in the proof of the following theorem which gives a new condition concerning the absolute continuity:

Theorem 1.3. *Let $V \in L^1_{loc}(\mathbb{R}^3)$ be a real periodic function with respect to the lattice $(2\pi\mathbb{Z})^3$. If V satisfies the conditions (1.2) and $|V| \in A_2(v)$ for some $v \in \mathbb{R}^3$, then the spectrum of $DAD^T + V(x)$ is purely absolutely continuous.*

Making use of resolvent estimates for a family of operators $(D + k)^2 + V$, $k \in \mathbb{C}^n$, Thomas [34] showed that the spectrum of the Schrödinger operator is purely absolutely continuous if $V \in L^2_{loc}(\mathbb{R}^3)$. Based on this approach, the absolute continuity of periodic operators has been extensively studied by many authors ([20, 7, 11, 12, 2, 3, 4, 19, 30, 25, 26, 27, 28]). Among others, Shen [25] established the absolute continuity of (1.1) for $V \in L^{n/2}_{loc}(\mathbb{R}^n)$, $n \geq 3$, which is best possible in the context of L^p potentials. This result was later extended by himself [27] to the Morrey class $\mathcal{M}^p(\mathbb{R}^n)$ with $p > (n-1)/2$, $n \geq 3$, under the smallness assumption (1.4). See also [26] for the Kato class in three dimensions.

Finally, we would like to emphasize that the class \mathcal{F} contains the global Kato (\mathcal{K}) and Rollnik (\mathcal{R}) classes which are defined by

$$\|f\|_{\mathcal{K}} := \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|f(y)|}{|x-y|} dy < \infty$$

and

$$\|f\|_{\mathcal{R}} := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|f(x)f(y)|}{|x-y|^2} dx dy < \infty,$$

respectively. These are fundamental ones in spectral and scattering theory (*cf.* [13, 29]), and their usefulness was recently revealed in the work of Rodnianski and Schlag [21] concerning dispersive properties of Schrödinger equations.

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.1 which says that the self-adjoint operator $DAD^T + V$ can be defined by a quadratic form under the condition (1.2). Based on the Thomas approach, we make use of the weighted L^2 resolvent estimates in Proposition 3.2 to prove Theorem 1.3 in Section 3. Sections 4 and 5 are devoted to proving Proposition 3.2. The key ingredient in the proof is the weighted L^2 spectral projection estimates of Proposition 4.1 in Section 4. These estimates will be shown in Section 5 by using Lemma 5.1 which is a weighted version of an oscillatory integral theorem of Stein.

From now on, we will use the letter C for positive constants that may be different at each occurrence.

2. SELF-ADJOINTNESS

In this section we prove Theorem 1.1. Namely, we will show that the operator $DAD^T + V$ is self-adjoint under the condition (1.2) on a real periodic V .

Let $\psi \in C_0^1(\mathbb{R}^3)$. From the definition of V_N , we see that

$$\int_{\mathbb{R}^3} |\psi|^2 |V| dx \leq \int_{\mathbb{R}^3} |\psi|^2 |V_N| dx + N \int_{\mathbb{R}^3} |\psi|^2 dx. \quad (2.1)$$

Now we claim that

$$\int_{\mathbb{R}^3} |\psi|^2 |V_N| dx \leq C \|V_N \chi_\Omega\|_{\mathcal{F}} \int_{\mathbb{R}^3} |\nabla \psi|^2 dx + C \int_{\Omega} |V_N| dx \int_{\mathbb{R}^3} |\psi|^2 dx. \quad (2.2)$$

Assuming this, we get from (2.1) that

$$\int_{\mathbb{R}^3} |\psi|^2 |V| dx \leq C \|V_N \chi_\Omega\|_{\mathcal{F}} \int_{\mathbb{R}^3} |\nabla \psi|^2 dx + \left(C \int_{\Omega} |V_N| dx + N \right) \int_{\mathbb{R}^3} |\psi|^2 dx.$$

Since $\langle A \nabla \psi, \nabla \psi \rangle \geq C |\nabla \psi|^2$ and $V \in L_{\text{loc}}^1(\mathbb{R}^3)$, by the condition (1.2), we conclude that

$$\int_{\mathbb{R}^3} |\psi|^2 |V| dx \leq C \varepsilon \int_{\mathbb{R}^3} \langle A \nabla \psi, \nabla \psi \rangle dx + C_N \int_{\mathbb{R}^3} |\psi|^2 dx \quad (2.3)$$

if N is sufficiently large. Hence, if ε is small enough so that $C \varepsilon < 1/2$, then (2.3) clearly implies that the symmetric quadratic form q given in (1.3) is semi-bounded

from below and closable on $H^1(\mathbb{R}^3)$. Thus, it defines a unique self-adjoint operator, which we denote by $DAD^T + V$, such that

$$q[f, g] = \int_{\mathbb{R}^3} \langle (DAD^T + V)f, g \rangle dx$$

for $f \in \text{Dom}[DAD^T + V]$ and $g \in H^1(\mathbb{R}^3)$. Also,

$$\text{Dom}[DAD^T + V] = \{f \in H^1(\mathbb{R}^3) : (DAD^T + V)f \in L^2(\mathbb{R}^3)\}.$$

Now it remains to show the claim (2.2). For this, we first note that

$$\left| \psi(y) - \frac{1}{|\Omega|} \int_{\Omega} \psi(x) dx \right| \leq CI_1(|\nabla \psi|_{\chi_{\Omega}})(y),$$

where I_1 is the fractional integral operator of order 1 given in (1.7) (see Lemma 7.16 in [9]). Then, by using this, $(a^{1/2} + b^{1/2})^2 \leq 2(a + b)$, and Hölder's inequality, it is not difficult to see that

$$\int_{\Omega} |\psi|^2 |V_N| dx \leq C \int_{\mathbb{R}^3} |I_1(|\nabla \psi|_{\chi_{\Omega}})|^2 |V_N|_{\chi_{\Omega}} dx + C \int_{\Omega} |V_N| dx \int_{\Omega} |\psi|^2 dx. \quad (2.4)$$

Now we will use the following lemma, which characterizes weighted L^2 inequalities for the fractional integral operator I_1 , due to Kerman and Sawyer [16] (see Theorem 2.3 there and also Lemma 2.1 in [1]):

Lemma 2.1. *Let w be a nonnegative measurable function on \mathbb{R}^3 . Then there exists a constant C_w depending on w such that*

$$\|I_1 f\|_{L^2(w)} \leq C_w \|f\|_{L^2} \quad (2.5)$$

for all measurable functions f on \mathbb{R}^3 if and only if

$$\|w\|_{\mathcal{F}} := \sup_Q \left(\int_Q w(x) dx \right)^{-1} \int_Q \int_Q \frac{w(x)w(y)}{|x-y|} dx dy < \infty.$$

Here the sup is taken over all dyadic cubes Q in \mathbb{R}^3 , and the constant C_w may be taken to be a constant multiple of $\|w\|_{\mathcal{F}}^{1/2}$.

Applying this lemma to the first term in the right-hand side of (2.4), we see that

$$\int_{\mathbb{R}^3} |I_1(|\nabla \psi|_{\chi_{\Omega}})|^2 |V_N|_{\chi_{\Omega}} dx \leq C \|V_N \chi_{\Omega}\|_{\mathcal{F}} \int_{\Omega} |\nabla \psi|^2 dx. \quad (2.6)$$

Combining (2.4) and (2.6), we now get

$$\int_{\Omega} |\psi|^2 |V_N| dx \leq C \|V_N \chi_{\Omega}\|_{\mathcal{F}} \int_{\Omega} |\nabla \psi|^2 dx + C \int_{\Omega} |V_N| dx \int_{\Omega} |\psi|^2 dx \quad (2.7)$$

which readily implies the claim (2.2) because V_N is periodic with respect to the cell Ω .

3. ABSOLUTE CONTINUITY

In this section we prove Theorem 1.3. For $k \in \mathbb{C}^3$, we need to define the operator $(D + k)A(D + k)^T + V$ on $L^2(\mathbb{T}^3)$, where $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3 \approx [0, 2\pi)^3 = \Omega$. For this, we first let

$$H^1(\mathbb{T}^3) = \left\{ \psi \in L^2(\Omega) : \psi(x) = \sum_{\mathbf{n} \in \mathbb{Z}^3} a_{\mathbf{n}} e^{i\mathbf{n} \cdot x} \text{ and } \sum_{\mathbf{n} \in \mathbb{Z}^3} |\mathbf{n}|^2 |a_{\mathbf{n}}|^2 < \infty \right\}.$$

Let us then consider the following quadratic form depending on k :

$$q(k)[\phi, \psi] = \int_{\Omega} \langle [(D + k)\phi]A, (D + \bar{k})\psi \rangle + \langle V\phi, \psi \rangle dx, \quad (3.1)$$

where $\phi, \psi \in H^1(\mathbb{T}^3)$ and \bar{k} denotes the conjugate of k . Now we observe that

$$\int_{\Omega} |\psi|^2 |V| dx \leq C\varepsilon \int_{\Omega} \langle A\nabla\psi, \nabla\psi \rangle dx + C_N \int_{\Omega} |\psi|^2 dx \quad (3.2)$$

if N is large. Indeed, from the definition of V_N ,

$$\int_{\Omega} |\psi|^2 |V| dx \leq \int_{\Omega} |\psi|^2 |V_N| dx + N \int_{\Omega} |\psi|^2 dx.$$

So, (3.2) follows from combining this, (2.7) and (1.2). If we choose ε in (3.2) small enough so that $C\varepsilon < 1/2$, then this implies that the quadratic form $q(k)$ is strictly m-sectorial. Thus, there exists a unique closed operator, which we denote by $(D + k)A(D + k)^T + V$, such that

$$q(k)[\phi, \psi] = \int_{\Omega} \langle [(D + k)A(D + k)^T + V]\phi, \psi \rangle dx$$

for $\phi \in \text{Dom}[(D + k)A(D + k)^T + V]$ and $\psi \in H^1(\mathbb{T}^3)$. (See [14] for details.) Also,

$$\begin{aligned} \text{Dom}[(D + k)A(D + k)^T + V] &= \{\phi \in H^1(\mathbb{T}^3) : [(D + k)A(D + k)^T + V]\phi \in L^2(\mathbb{T}^3)\} \\ &= \{\phi \in H^1(\mathbb{T}^3) : (DAD^T + V)\phi \in L^2(\mathbb{T}^3)\} \end{aligned}$$

is independent of k .

Next we choose $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ such that

$$|\mathbf{a}| = 1 \quad \text{and} \quad \mathbf{a}A = (s_0, 0, 0), \quad s_0 > 0, \quad (3.3)$$

and let

$$L = \{\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3 : |\mathbf{b}| < \sqrt{3} \text{ and } \langle \mathbf{b}, \mathbf{a} \rangle = 0\}. \quad (3.4)$$

For a fixed $\mathbf{b} \in L$, let us now consider a family of operators

$$H_V(\lambda) = (D + \lambda\mathbf{a} + \mathbf{b})A(D + \lambda\mathbf{a} + \mathbf{b})^T + V, \quad \lambda \in \mathbb{C},$$

defined by the quadratic form (3.1). Then the following lemma is standard (see Propositions 4.5 and 3.9 in [26] and [27], respectively).

Lemma 3.1. *If, for every $\mathbf{b} \in L$, the family of operators $\{H_V(\lambda) : \lambda \in \mathbb{C}\}$ has no common eigenvalue, then the spectrum of the self-adjoint operator $DAD^T + V$ is purely absolutely continuous.*

To prove Theorem 1.3, by this lemma we only need to show that $\{H_V(\lambda) : \lambda \in \mathbb{C}\}$ has no common eigenvalue. For this, let us first consider $\lambda = \delta_0 + i\rho$, where

$$\delta_0 = \frac{1}{a_1} \left(\frac{1}{2} - b_1 \right).$$

Here a_1 and b_1 are given in (3.3) and (3.4), respectively. Since $\langle \mathbf{a}A, \mathbf{a} \rangle = a_1 s_0$ and $|\mathbf{a}| = 1$, $a_1 \neq 0$. From now on, we will show that $\{H_V(\delta_0 + i\rho) : \rho \in \mathbb{R}\}$ has no common eigenvalue. This will be based on the following weighted L^2 resolvent estimates which will be obtained in the next section.

Proposition 3.2. *Let $w \in L^1(\mathbb{T}^3)$ and $w \in A_2(v)$ for some $v \in \mathbb{R}^3$. Assume that $w \geq c_w$ for some constant $c_w > 0$ and*

$$\|w\|_{\mathcal{F}(\mathbb{T}^3)} := \sup_{z \in \Omega, 0 < r < 4\pi} \left(\int_{Q(z,r)} w(x) dx \right)^{-1} \int_{Q(z,r)} \int_{Q(z,r)} \frac{w(x)w(y)}{|x-y|} dx dy < \infty.$$

Then, if $\psi \in H^1(\mathbb{T}^3)$ and $H_0(\delta_0 + i\rho)\psi \in L^2(\mathbb{T}^3, w^{-1}dx)$, we have for $|\rho| \geq 1$

$$\|\psi\|_{L^2(\mathbb{T}^3, w dx)} \leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)} \|H_0(\delta_0 + i\rho)\psi\|_{L^2(\mathbb{T}^3, w^{-1}dx)}, \quad (3.5)$$

where C is a constant independent of ρ and c_w .

Remark 3.3. The estimate (3.5) is a uniform Sobolev inequality on the torus \mathbb{T}^3 for the second-order elliptic operator $H_0(\delta_0 + i\rho)$. Similar inequalities were obtained in the setting of \mathbb{R}^n by many authors ([15, 5, 6, 23]) to study unique continuation properties of differential operators. Also, (3.5) was shown in [27] for w in the Morrey class.

Now we suppose that E is an eigenvalue for $H_V(\lambda)$ for all $\lambda \in \mathbb{C}$. Then there exists $\psi_\rho \in \text{Dom}[H_V(\lambda)]$ particularly for $\lambda = \delta_0 + i\rho$ such that $\|\psi_\rho\|_{L^2(\mathbb{T}^3)} = 1$ and $H_V(\lambda)\psi_\rho = E\psi_\rho$. By the condition (1.2), we can choose N so large that

$$\sup_{z \in \Omega, 0 < r < 4\pi} \left(\int_{Q(z,r)} |V_N(x)| dx \right)^{-1} \int_{Q(z,r)} \int_{Q(z,r)} \frac{|V_N(x)V_N(y)|}{|x-y|} dx dy < \varepsilon.$$

Let us now consider $w(x) = |V_N(x)| + f_\delta(x)$, where f_δ is a periodic function with respect to $(2\pi\mathbb{Z})^3$, which is given by $f_\delta(x) = \delta/|x|^2$ with $\delta > 0$ for $x \in \Omega$. Then, w is periodic with respect to $(2\pi\mathbb{Z})^3$ and $w \geq \tilde{c} > 0$ for some constant \tilde{c} . Recall from Remark 1.2 that $\|f_\delta \chi_\Omega\|_{\mathcal{F}} \leq C \|f_\delta \chi_\Omega\|_{\mathcal{M}^p(\mathbb{R}^3)} \leq C\delta$ and that the norm $\|\cdot\|_{\mathcal{F}}$ is the least bound for

$$\int |I_1 f(x)|^2 w(x) dx \leq C \|w\|_{\mathcal{F}} \int |f(x)|^2 dx.$$

It is now clear that

$$\|w_1 + w_2\|_{\mathcal{F}} \leq C(\|w_1\|_{\mathcal{F}} + \|w_2\|_{\mathcal{F}}). \quad (3.6)$$

Since $\|w\|_{\mathcal{F}(\mathbb{T}^3)} = C\|w \chi_\Omega\|_{\mathcal{F}}$ (see the paragraph below (1.2)), from (3.6) we can take δ small enough so that

$$\sup_{z \in \Omega, 0 < r < 4\pi} \left(\int_{Q(z,r)} w(x) dx \right)^{-1} \int_{Q(z,r)} \int_{Q(z,r)} \frac{w(x)w(y)}{|x-y|} dx dy < \varepsilon. \quad (3.7)$$

Using Lemma 2.1 together with (3.7), we now get

$$\begin{aligned}
\int_{\Omega} |V\psi_{\rho}|^2 w^{-1} dx &\leq N^2 \int_{\Omega \cap \{V(x) \leq N\}} |\psi_{\rho}|^2 w^{-1} dx + \int_{\Omega \cap \{V(x) > N\}} |\psi_{\rho}|^2 |V_N|^2 w^{-1} dx \\
&\leq N^2 \int_{\Omega} |\psi_{\rho}|^2 w^{-1} dx + \int_{\Omega} |\psi_{\rho}|^2 w dx \\
&\leq N^2 \tilde{c}^{-1} \int_{\Omega} |\psi_{\rho}|^2 dx + C\varepsilon \int_{\Omega} |\nabla \psi_{\rho}|^2 dx.
\end{aligned} \tag{3.8}$$

Hence, $V\psi_{\rho} \in L^2(\mathbb{T}^3, w^{-1} dx)$ because $\psi_{\rho} \in H^1(\mathbb{T}^3)$. Also, $\psi_{\rho} \in L^2(\mathbb{T}^3, w^{-1} dx)$ since $w \geq \tilde{c} > 0$. Since $H_V(\lambda)\psi_{\rho} = E\psi_{\rho}$, we now conclude that

$$H_0(\delta_0 + i\rho)\psi_{\rho} = E\psi_{\rho} - V\psi_{\rho} \in L^2(\mathbb{T}^3, w^{-1} dx),$$

and from (3.8)

$$\|H_0(\delta_0 + i\rho)\psi_{\rho}\|_{L^2(\mathbb{T}^3, w^{-1} dx)} \leq (N + |E|)\|\psi_{\rho}\|_{L^2(\mathbb{T}^3, w^{-1} dx)} + \|\psi_{\rho}\|_{L^2(\mathbb{T}^3, w dx)}. \tag{3.9}$$

On the other hand, by Proposition 3.2 with (3.7), we see that for $\rho \geq 1$

$$\|\psi_{\rho}\|_{L^2(\mathbb{T}^3, w dx)} \leq C\varepsilon \|H_0(\delta_0 + i\rho)\psi_{\rho}\|_{L^2(\mathbb{T}^3, w^{-1} dx)}.$$

So if ε is chosen so small that $C\varepsilon \leq 1/2$, from (3.9) we have

$$\|H_0(\delta_0 + i\rho)\psi_{\rho}\|_{L^2(\mathbb{T}^3, w^{-1} dx)} \leq 2(N + |E|)\|\psi_{\rho}\|_{L^2(\mathbb{T}^3, w^{-1} dx)}. \tag{3.10}$$

Now we recall from Theorem 3.13 in [27] that there exists $c_{\rho} > 0$ such that $c_{\rho} \rightarrow \infty$ as $|\rho| \rightarrow \infty$ and

$$c_{\rho} c_w^{1/2} \|\psi\|_{L^2(\mathbb{T}^3, w^{-1} dx)} \leq \|w\|_{L^1(\mathbb{T}^3)}^{1/2} \|H_0(\delta_0 + i\rho)\psi\|_{L^2(\mathbb{T}^3, w^{-1} dx)}$$

if $\psi \in H^1(\mathbb{T}^3)$, $H_0(\delta_0 + i\rho)\psi \in L^2(\mathbb{T}^3, w^{-1} dx)$, and $w \in L^1(\mathbb{T}^3)$ with $w \geq c_w > 0$. By combining this and (3.10), it follows that

$$c_{\rho} c_w^{1/2} \leq \|w\|_{L^1(\mathbb{T}^3)}^{1/2} 2(N + |E|) < \infty.$$

This leads to a contradiction since $c_{\rho} \rightarrow \infty$ as $\rho \rightarrow \infty$. Thus $\{H_V(z) : z \in \mathbb{C}\}$ has no common eigenvalue, and so Theorem 1.3 is proved by Lemma 3.1.

4. WEIGHTED L^2 RESOLVENT ESTIMATES

This section is devoted to proving Proposition 3.2. By an elementary rotation argument, we may first assume that $w \in A_2(\mathbb{R})$ in the x_1 variable uniformly in other variables $x' = (x_2, x_3) \in \mathbb{T}^2$. Then we need to show that for $\psi \in L^2(\mathbb{T}^3, w^{-1} dx)$,

$$\|H_0(\delta_0 + i\rho)^{-1}\psi\|_{L^2(\mathbb{T}^3, w dx)} \leq C\|w\|_{\mathcal{F}(\mathbb{T}^3)}\|\psi\|_{L^2(\mathbb{T}^3, w^{-1} dx)}, \tag{4.1}$$

where

$$H_0(\delta_0 + i\rho)^{-1}\psi(x) = \sum_{\mathbf{n}=(n_1, n_2, n_3) \in \mathbb{Z}^3} \frac{\widehat{\psi}(\mathbf{n})e^{i\mathbf{n} \cdot x}}{(\mathbf{n} + k)A(\mathbf{n} + k)^T}$$

and $k = (\delta_0 + i\rho)\mathbf{a} + \mathbf{b}$ with \mathbf{a}, \mathbf{b} given as in (3.3), (3.4). To show (4.1), we first decompose ψ as $\psi = \sum_{j=-\infty}^{\infty} \psi_j$, where

$$\begin{aligned}\psi_j &= \sum_{n_1 \in [2^{j-1}, 2^j - 1]} \widehat{\psi}(\mathbf{n}) e^{i\mathbf{n} \cdot \mathbf{x}} \quad \text{for } j \geq 1, \\ \psi_j &= \sum_{n_1 \in [-2^{-j+1}, -2^{-j-1}]} \widehat{\psi}(\mathbf{n}) e^{i\mathbf{n} \cdot \mathbf{x}} \quad \text{for } j \leq -1,\end{aligned}$$

and

$$\psi_0 = \sum_{n_1=0} \widehat{\psi}(\mathbf{n}) e^{i\mathbf{n} \cdot \mathbf{x}}.$$

Then by the Littlewood-Paley theory on \mathbb{T}^1 in weighted L^2 spaces (see Chap. XV in [36] and also [17]), it is enough to show (4.1) for ψ_j uniformly in j . Indeed, if we have

$$\|H_0(\delta_0 + i\rho)^{-1} \psi_j\|_{L^2(\mathbb{T}^3, w dx)} \leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)} \|\psi_j\|_{L^2(\mathbb{T}^3, w^{-1} dx)} \quad (4.2)$$

uniformly in j , then by the Littlewood-Paley theory and the condition $w \in A_2(\mathbb{R})$, we get

$$\begin{aligned}\|H_0(\delta_0 + i\rho)^{-1} \psi\|_{L^2(\mathbb{T}^3, w dx)} &\leq C \left\| \left(\sum_{j \in \mathbb{Z}} |H_0(\delta_0 + i\rho)^{-1} \psi_j|^2 \right)^{1/2} \right\|_{L^2(\mathbb{T}^3, w dx)} \\ &\leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)} \left\| \left(\sum_{j \in \mathbb{Z}} |\psi_j|^2 \right)^{1/2} \right\|_{L^2(\mathbb{T}^3, w^{-1} dx)} \\ &\leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)} \|\psi\|_{L^2(\mathbb{T}^3, w^{-1} dx)}\end{aligned}$$

as desired.

From now on, we will show (4.2) only for the case $j \geq 1$ because the other case $j \leq 0$ can be shown in the same way. First we recall from (6.5) in [27] (see also (3.5) in [25]) that

$$(\mathbf{n} + k)A(\mathbf{n} + k)^T = |(\mathbf{n} + \mathbf{b})B|^2 + 2\delta_0(n_1 + b_1)s_0 + (\delta_0^2 - \rho^2)a_1s_0 + 2i\rho(n_1 + \frac{1}{2})s_0$$

where B is a 3×3 symmetric, positive definite matrix such that $A = B^2$ (i.e., $B = \sqrt{A}$). In fact, this follows easily from (3.3) and (3.4). Fix $j \geq 1$. In view of the fact that $n_1 + 1/2 \sim 2^j$, we let $z_j = -\rho^2 a_1 s_0 + 2i\rho 2^j s_0$, and we consider the following operator

$$((D + \mathbf{b})A(D + \mathbf{b})^T + z_j)^{-1} \psi = \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{\widehat{\psi}(\mathbf{n}) e^{i\mathbf{n} \cdot \mathbf{x}}}{|(\mathbf{n} + \mathbf{b})B|^2 + z_j}.$$

Now we will show that

$$\left\| \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{\widehat{\psi}_j(\mathbf{n}) e^{i\mathbf{n} \cdot \mathbf{x}}}{|(\mathbf{n} + \mathbf{b})B|^2 + z_j} \right\|_{L^2(\mathbb{T}^3, w dx)} \leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)} \|\psi_j\|_{L^2(\mathbb{T}^3, w^{-1} dx)} \quad (4.3)$$

and

$$\begin{aligned} \left\| H_0(\delta_0 + \rho)^{-1} \psi_j - \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{\widehat{\psi_j}(\mathbf{n}) e^{i\mathbf{n} \cdot x}}{|(\mathbf{n} + \mathbf{b})B|^2 + z_j} \right\|_{L^2(\mathbb{T}^3, w dx)} \\ \leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)} \|\psi_j\|_{L^2(\mathbb{T}^3, w^{-1} dx)}. \end{aligned} \quad (4.4)$$

Then the desired estimate (4.2) follows directly from combining (4.3) and (4.4), and so the proof of Proposition 3.2 is completed.

To show the first estimate (4.3), we consider the family of operators

$$S_\xi \psi(x) = \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{\widehat{\psi_j}(\mathbf{n}) e^{i\mathbf{n} \cdot x}}{[(\mathbf{n} + \mathbf{b})A(\mathbf{n} + \mathbf{b})^T + z_j]^\xi}, \quad \xi \in \mathbb{C},$$

where

$$\begin{aligned} \operatorname{Re} \sqrt{z_j} &= |z_j|^{1/2} \cos\left(\frac{1}{2} \arg(z_j)\right) \geq \frac{1}{2} \frac{|\operatorname{Im} z_j|}{|z_j|^{1/2}} \\ &\geq \frac{c|\rho|2^j}{|\rho| + \sqrt{|\rho|2^j}} \\ &\geq c \min(2^j, \sqrt{|\rho|2^j}) \geq c_0 > 0. \end{aligned}$$

Then we have

$$S_\xi \psi(x) = \int_{\Omega} G_\xi(x - y) \psi(y) dy$$

and the integral kernel G_ξ of S_ξ satisfies

$$|G_\xi(x)| \leq C e^{c|\operatorname{Im} \xi|} \left(1 + \sum_{|x + 2\pi \mathbf{n}| \leq C} \frac{1}{|x + 2\pi \mathbf{n}|} \right)$$

for $\operatorname{Re} \xi = 1$. See (6.10) in [25]. It follows now that

$$\begin{aligned} |S_\xi \psi(x)| &\leq C e^{c|\operatorname{Im} \xi|} \left(\int_{\Omega} |\psi(y)| dy + \int_{\Omega} \sum_{|x - y + 2\pi \mathbf{n}| \leq C} \frac{\psi(y)}{|x - y + 2\pi \mathbf{n}|} dy \right) \\ &\leq C e^{c|\operatorname{Im} \xi|} \left(\int_{\Omega} |\psi(y)| dy + I_2(|\psi| \chi_{\Omega'})(x) \right), \end{aligned}$$

where $\Omega' = \bigcup_{|\mathbf{n}| \leq C} (\Omega + 2\pi \mathbf{n})$ and I_2 is the fractional integral operator of order 2 given in (1.7). Here, for the last inequality we used the fact that ψ is periodic. Hence, for $\operatorname{Re} \xi = 1$

$$\begin{aligned} &|w^{\xi/2} S_\xi(w^{\xi/2} \psi)(x)| \\ &\leq C e^{c|\operatorname{Im} \xi|} \left(w(x)^{1/2} \int_{\Omega} |\psi(y)| w(y)^{1/2} dy + w(x)^{1/2} I_2(|\psi| \chi_{\Omega'} w^{1/2})(x) \right) \\ &:= C e^{c|\operatorname{Im} \xi|} (I + II). \end{aligned} \quad (4.5)$$

Then, using Hölder's inequality, we can bound the first term in the right-hand side of (4.5) as

$$\begin{aligned} \|I\|_{L^2(\Omega, dx)} &\leq \int_{\Omega} |\psi(y)| w(y)^{1/2} dy \left(\int_{\Omega} w(x) dx \right)^{1/2} \\ &\leq \|\psi\|_{L^2(\Omega)} \int_{\Omega} w(x) dx \\ &\leq C \|\chi_{\Omega} w\|_{\mathcal{F}} \|\psi\|_{L^2(\Omega)} \\ &\leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)} \|\psi\|_{L^2(\Omega)}. \end{aligned}$$

Here, for the last inequality we used the fact that w is periodic. Since $1/|x-y| \geq c > 0$ for $x, y \in \Omega$, we see here that

$$\begin{aligned} c \int_{\Omega} w(x) dx &\leq \left(\int_{\Omega} w(x) dx \right)^{-1} \int_{\Omega} \int_{\Omega} \frac{w(x)w(y)}{|x-y|} dx dy \\ &\leq \sup_{z \in \mathbb{R}^3, r > 0} \left(\int_{Q(z, r) \cap \Omega} w(x) dx \right)^{-1} \int_{Q(z, r) \cap \Omega} \int_{Q(z, r) \cap \Omega} \frac{w(x)w(y)}{|x-y|} dx dy \\ &= \|\chi_{\Omega} w\|_{\mathcal{F}}. \end{aligned}$$

On the other hand, for the second term we will use the following estimate

$$\|I_2 f\|_{L^2(w)} \leq C \|w\|_{\mathcal{F}} \|f\|_{L^2(w^{-1})}, \quad (4.6)$$

which follows by combining (2.5) in Lemma 2.1 and its dual estimate

$$\|I_1 f\|_{L^2} \leq C_w \|f\|_{L^2(w^{-1})}$$

since $I_2 = I_1 I_1$. Using (4.6), we now see that

$$\begin{aligned} \|II\|_{L^2(\Omega, dx)} &\leq \|\chi_{\Omega'} w^{1/2} I_2(|\psi| \chi_{\Omega'} w^{1/2})\|_{L^2} \\ &\leq C \|\chi_{\Omega'} w\|_{\mathcal{F}} \|\psi \chi_{\Omega'}\|_{L^2} \\ &\leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)} \|\psi\|_{L^2(\Omega)}. \end{aligned}$$

Here, for the last inequality we used the fact that ψ and w are periodic. Consequently, we get

$$\|S_1 \psi\|_{L^2(\Omega, w dx)} \leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)} \|\psi\|_{L^2(\Omega, w^{-1} dx)}$$

and (4.3) is now proved.

It remains to show the second estimate (4.4). First we write

$$\begin{aligned} H_0(\delta_0 + \rho)^{-1} \psi_j &- \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{\widehat{\psi_j}(\mathbf{n}) e^{i\mathbf{n} \cdot \mathbf{x}}}{|(\mathbf{n} + \mathbf{b})B|^2 + z_j} \\ &= \sum_{M=1}^{\infty} \sum_{\{\mathbf{n} \in \mathbb{Z}^3 : M-1 \leq |\mathbf{n}B| < M\}} \frac{\widehat{\psi_j}(\mathbf{n}) e^{i\mathbf{n} \cdot \mathbf{x}} [|(\mathbf{n} + \mathbf{b})B|^2 + z_j - (\mathbf{n} + k)A(\mathbf{n} + k)^T]}{[(\mathbf{n} + k)A(\mathbf{n} + k)^T] [|(\mathbf{n} + \mathbf{b})B|^2 + z_j]}. \end{aligned} \quad (4.7)$$

We then consider the second-order elliptic operator DAD^T on the torus $[0, 2\pi]^3 \approx \mathbb{R}^3 / (2\pi\mathbb{Z})^3$ which has a complete set of eigenfunctions $\{e^{i\mathbf{n} \cdot \mathbf{x}} : \mathbf{n} \in \mathbb{Z}^3\}$ with the

corresponding eigenvalues $\{\mathbf{n}A\mathbf{n}^T : \mathbf{n} \in \mathbb{Z}^3\}$. Hence,

$$P_M\psi = \sum_{\{\mathbf{n} \in \mathbb{Z}^3 : \mathbf{n}A\mathbf{n}^T \in [(M-1)^2, M^2]\}} \widehat{\psi}(\mathbf{n})e^{i\mathbf{n} \cdot \mathbf{x}} = \sum_{\{\mathbf{n} \in \mathbb{Z}^3 : |\mathbf{n}B| \in [M-1, M]\}} \widehat{\psi}(\mathbf{n})e^{i\mathbf{n} \cdot \mathbf{x}}$$

is the projection of ψ to the subspace of $L^2(\Omega)$, spanned by eigenfunctions with eigenvalues in $[(M-1)^2, M^2]$. In view of this fact, the following estimate for the spectral projection, obtained by Sogge [31] (see Theorem 2.2 (i) there), can be used to bound the right-hand side of (4.7) in L^2 space: For $\psi \in L^2(\Omega)$ and $1 \leq p \leq 4/3$,

$$\|P_M\psi\|_{L^2} \leq CM^{\frac{1}{2}(\frac{6}{p}-4)}\|\psi\|_{L^p}. \quad (4.8)$$

See the proof of Lemma 5.2 in [25]. This spectral projection estimate can be thought of as a discrete version of the Fourier restriction estimate of Stein and Tomas [35]. A weighted version of (4.8),

$$\|P_M\psi\|_{L^2(\mathbb{T}^3, wdx)} \leq CM^{1/2}\|w\|_{\mathcal{M}^p(\mathbb{T}^3)}\|\psi\|_{L^2(\mathbb{T}^3)}, \quad 1 < p \leq 3/2, \quad (4.9)$$

can be also found in Section 5 of [27] where it was used in handling (4.7) in the weighted L^2 space with the weight w in the Morrey space $\mathcal{M}^p(\mathbb{R}^3)$. Note that (4.9) is the analog of the weighted L^2 Fourier restriction estimate obtained in [5, 6]. In our case, we need the following lemma which can be viewed as an extension of (4.9) to the class \mathcal{F} because $\mathcal{M}^p(\mathbb{R}^3) \subset \mathcal{F}$.

Proposition 4.1. *Let $w \in L^1(\mathbb{T}^3)$. Assume that $w \geq c_w > 0$ and $\|w\|_{\mathcal{F}(\mathbb{T}^3)} < \infty$. Then one has*

$$\left\| \sum_{|\mathbf{n}B| \in [k-1, k]} \widehat{\psi}(\mathbf{n})e^{i\mathbf{n} \cdot \mathbf{x}} \right\|_{L^2(\mathbb{T}^3, wdx)} \leq Ck^{1/2}\|w\|_{\mathcal{F}(\mathbb{T}^3)}^{1/2}\|\psi\|_{L^2(\mathbb{T}^3)} \quad (4.10)$$

for $\psi \in L^2(\mathbb{T}^3)$ and $k \geq 1$. Here, $B = \sqrt{A} \geq 0$ and C depends only on A .

Using (4.7), Minkowski's inequality, and this proposition which will be shown in the next section, the left-hand side of (4.4) is now bounded by

$$\begin{aligned} & C\|w\|_{\mathcal{F}(\mathbb{T}^3)}^{1/2} \sum_{M=1}^{\infty} M^{1/2} \\ & \times \left\| \sum_{|\mathbf{n}B| \in [M-1, M]} \frac{\widehat{\psi}_j(\mathbf{n})e^{i\mathbf{n} \cdot \mathbf{x}} [|(\mathbf{n} + \mathbf{b})B|^2 + z_j - (\mathbf{n} + \mathbf{k})A(\mathbf{n} + \mathbf{k})^T] }{ [(\mathbf{n} + \mathbf{k})A(\mathbf{n} + \mathbf{k})^T] [|(\mathbf{n} + \mathbf{b})B|^2 + z_j] } \right\|_{L^2(\mathbb{T}^3)}. \end{aligned} \quad (4.11)$$

Then we recall from (5.10) in [25] that

$$\begin{aligned} & \sum_{M=1}^{\infty} M \sup \left| \frac{|(\mathbf{n} + \mathbf{b})B|^2 + z_j - (\mathbf{n} + \mathbf{k})A(\mathbf{n} + \mathbf{k})^T|}{|(\mathbf{n} + \mathbf{k})A(\mathbf{n} + \mathbf{k})^T| [|(\mathbf{n} + \mathbf{b})B|^2 + z_j] } \right| \\ & \leq \sum_{M=1}^{\infty} M \sup \frac{|\rho|2^j}{(|(\mathbf{n} + \mathbf{b})B|^2 - \rho^2 a_1 s_0| + 2^j |\rho|)^2} \leq C. \end{aligned}$$

Here, the sup is taken over all $\mathbf{n} \in \mathbb{Z}^3$ such that $n_1 \in [2^{j-1}, 2^j - 1]$ and $|\mathbf{n}B| \in [M-1, M)$. Using this and the dual estimate

$$\left\| \sum_{|\mathbf{n}B| \in [k-1, k)} \widehat{\psi}(\mathbf{n}) e^{i\mathbf{n} \cdot x} \right\|_{L^2(\mathbb{T}^3)} \leq Ck^{1/2} \|w\|_{\mathcal{F}(\mathbb{T}^3)}^{1/2} \|\psi\|_{L^2(\mathbb{T}^3, w^{-1} dx)}$$

of (4.10), (4.11) is bounded as follows:

$$\begin{aligned} (4.11) &\leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)}^{1/2} \sum_{M=1}^{\infty} M^{1/2} \left\| \sum_{|\mathbf{n}B| \in [M-1, M)} \widehat{\psi_j}(\mathbf{n}) e^{i\mathbf{n} \cdot x} \right\|_{L^2(\mathbb{T}^3)} \\ &\quad \times \sup \left| \frac{|(\mathbf{n} + \mathbf{b})B|^2 + z_j - (\mathbf{n} + k)A(\mathbf{n} + k)^T}{[(\mathbf{n} + k)A(\mathbf{n} + k)^T][(\mathbf{n} + \mathbf{b})B|^2 + z_j]} \right| \\ &\leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)} \|\psi_j\|_{L^2(\mathbb{T}^3, w^{-1} dx)} \\ &\quad \times \sum_{M=1}^{\infty} M \sup \left| \frac{|(\mathbf{n} + \mathbf{b})B|^2 + z_j - (\mathbf{n} + k)A(\mathbf{n} + k)^T}{[(\mathbf{n} + k)A(\mathbf{n} + k)^T][(\mathbf{n} + \mathbf{b})B|^2 + z_j]} \right| \\ &\leq C \|w\|_{\mathcal{F}(\mathbb{T}^3)} \|\psi_j\|_{L^2(\mathbb{T}^3, w^{-1} dx)}. \end{aligned}$$

Hence we get (4.4).

5. PROOF OF PROPOSITION 4.1

To show (4.10) in Proposition 4.1, we first let

$$\phi(x) = \sum_{|\mathbf{n}B| \in [k-1, k)} \widehat{\psi}(\mathbf{n}) e^{i\mathbf{n} \cdot x}$$

for $\psi \in L^2(\mathbb{T}^3)$. Then it is clear that $\phi \in C^\infty(\mathbb{T}^3)$, and

$$\|(DAD^T - k^2 - 2ik)\phi\|_{L^2(\mathbb{T}^3)} \leq Ck \|\psi\|_{L^2(\mathbb{T}^3)} \quad (5.1)$$

since ϕ is the projection of ψ to the subspace of $L^2(\Omega)$, spanned by eigenfunctions with eigenvalues in $[(k-1)^2, k^2)$. Hence, if we show the following uniform Sobolev inequality

$$\|\varphi\|_{L^2(\mathbb{T}^3, w dx)} \leq Ck^{-1/2} \|w\|_{\mathcal{F}(\mathbb{T}^3)}^{1/2} \|(DAD^T - k^2 - 2ik)\varphi\|_{L^2(\mathbb{T}^3)} \quad (5.2)$$

for $\varphi \in C^\infty(\mathbb{T}^3)$ and $k \geq 1$, then it follows from (5.1) that

$$\begin{aligned} \|\phi\|_{L^2(\mathbb{T}^3, w dx)} &\leq Ck^{-1/2} \|w\|_{\mathcal{F}(\mathbb{T}^3)}^{1/2} \|(DAD^T - k^2 - 2ik)\phi\|_{L^2(\mathbb{T}^3)} \\ &\leq Ck^{1/2} \|w\|_{\mathcal{F}(\mathbb{T}^3)}^{1/2} \|\psi\|_{L^2(\mathbb{T}^3)} \end{aligned}$$

as desired.

Now we have to show (5.2) which can be thought of as an extension of that in [27] for the Morrey class $\mathcal{M}^p(\mathbb{R}^3)$ to the class \mathcal{F} . Fix $x_0 \in \Omega$. Let $\tilde{\eta} \in C_0^\infty(Q(x_0, 1/2))$ be such that $\tilde{\eta} = 1$ on $Q(x_0, 1/4)$. Here, $Q(x, r)$ denotes the cube centered at x with side length r . Then we note that

$$\varphi(x) \tilde{\eta}(x)^2 = \tilde{\eta}(x) \int_{\mathbb{R}^3} F_{z,B}(x-y) (DAD^T + z)(\varphi \tilde{\eta})(y) dy, \quad (5.3)$$

where $F_{z,B}(x)$ is the Fourier transform of $(|yB|^2 + z)^{-1}$, given by

$$\begin{aligned} F_{z,B}(x) &= \frac{1}{\det(B)} \int_{\mathbb{R}^3} \frac{e^{-ixB^{-1} \cdot y}}{|y|^2 + z} dy \\ &= \frac{c}{\det(B)} \left(\frac{z}{|xB^{-1}|} \right)^{\frac{1}{2}(\frac{3}{2}-1)} K_{\frac{3}{2}-1}(\sqrt{z}|xB^{-1}|). \end{aligned}$$

Here, $K_{\frac{3}{2}-1}$ denotes the modified Bessel function of the third kind of order $3/2 - 1$ (see [18], p. 108). Since $x, y \in Q(x_0, 1/2)$, $|x - y| < 1$. From this, we rewrite the right-hand side of (5.3) as

$$\tilde{\eta}(x) \int_{\mathbb{R}^3} F_{z,B}(x - y) \eta(|x - y|) [(DAD^T + z)\varphi \cdot \tilde{\eta} - 2D\varphi A(D\tilde{\eta})^T - \varphi DAD^T \tilde{\eta}](y) dy,$$

where $\eta \in C_0^\infty((-2, 2))$ and $\eta(r) = 1$ if $|r| \leq 1$. Now we assume for the moment that

$$\left\| \int_{\mathbb{R}^3} F_{z,B}(x - y) \eta(|x - y|) f(y) dy \right\|_{L^2(\mathbb{R}^3, w dx)} \leq \frac{C}{|z|^{1/4}} \|w\|_{\mathcal{F}(\mathbb{R}^3)}^{1/2} \|f\|_{L^2(\mathbb{R}^3)}, \quad (5.4)$$

where $\operatorname{Re} \sqrt{z} \geq 1$. Using this, we then see that for $z = -(k + i)^2$

$$\begin{aligned} &\|\varphi \tilde{\eta}^2\|_{L^2(\mathbb{R}^3, w dx)} \\ &\leq Ck^{-1/2} \|w \tilde{\eta}^2\|_{\mathcal{F}(\mathbb{R}^3)}^{1/2} \\ &\quad \times (\|(DAD^T + z)\varphi \cdot \tilde{\eta}\|_{L^2(\mathbb{R}^3)} + \|D\varphi A(D\tilde{\eta})^T\|_{L^2(\mathbb{R}^3)} + \|\varphi DAD^T \tilde{\eta}\|_{L^2(\mathbb{R}^3)}) \\ &\leq Ck^{-1/2} \|w\|_{\mathcal{F}(\mathbb{T}^3)}^{1/2} \\ &\quad \times (\|(DAD^T - k^2 - 2ik)\varphi\|_{L^2(\mathbb{T}^3)} + \|D\varphi\|_{L^2(\mathbb{T}^3)} + \|\varphi\|_{L^2(\mathbb{T}^3)}), \end{aligned} \quad (5.5)$$

where we used for the last inequality the fact that w and ψ are periodic. We also see that for $k \geq 1$

$$\|D\varphi\|_{L^2(\mathbb{T}^3)} + \|\varphi\|_{L^2(\mathbb{T}^3)} \leq C\|(DAD^T - k^2 - 2ik)\varphi\|_{L^2(\mathbb{T}^3)}$$

using the Fourier series and Parseval's formula ([33]). Combining this and (5.5), we get

$$\|\varphi\|_{L^2(\mathbb{T}^3, w dx)} \leq Ck^{-1/2} \|w\|_{\mathcal{F}(\mathbb{T}^3)}^{1/2} \|(DAD^T - k^2 - 2ik)\varphi\|_{L^2(\mathbb{T}^3)}$$

as desired.

It remains to show (5.4). But this follows from the following lemma by using partition of unity and standard rescaling argument (cf. [27], Theorem 5.33). See also [31] (pp. 134-135) for the case of $L^2 \rightarrow L^q$ estimates.

Lemma 5.1. *Let $w \geq 0$ and $w \in \mathcal{F}(\mathbb{R}^3)$. Assume that $a \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and*

$$\operatorname{supp} a \subset \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : 1/2 \leq |x - y| \leq 2\}. \quad (5.6)$$

Then, for $f \in L^2(\mathbb{R}^3)$,

$$\left\| \int_{\mathbb{R}^3} e^{i\lambda|x-y|} a(x, y) f(y) dy \right\|_{L^2(\mathbb{R}^3, w dx)} \leq C|\lambda|^{-1/2} \|w\|_{\mathcal{F}(\mathbb{R}^3)}^{1/2} \|f\|_{L^2(\mathbb{R}^3)}, \quad (5.7)$$

where $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$.

Remark 5.2. This lemma is an extension of Theorem 5.5 in [27] for the Morrey class $\mathcal{M}^p(\mathbb{R}^3)$ to the class \mathcal{F} , which is also a weighted version of an oscillatory integral theorem of Stein (see [32], p. 380).

Proof of Lemma 5.1. Using partition of unity and the assumption (5.6), we may first assume that

$$\text{supp } a \subset \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x - x_0| < \delta, |y - y_0| < \delta, 1/2 \leq |x - y| \leq 2\}$$

for some $x_0, y_0 \in \mathbb{R}^3$ and a sufficiently small $\delta > 0$. Since $1/|x|^2 \in \mathcal{M}^p(\mathbb{R}^3) \subset \mathcal{F}$ clearly, we may also assume that $w > 0$ by replacing w with $\tilde{w}(x) = w(x) + \varepsilon/|x|^2$ and then letting $\varepsilon \rightarrow 0$. By duality, (5.7) is equivalent to

$$\left\| \int_{\mathbb{R}^3} e^{i\lambda|x-y|} a(x, y) f(y) dy \right\|_{L^2(\mathbb{R}^3)} \leq C |\lambda|^{-1/2} \|w\|_{\mathcal{F}(\mathbb{R}^3)}^{1/2} \|f\|_{L^2(\mathbb{R}^3, w^{-1} dx)}.$$

By translation we may assume here that $x_0 = 0$, and since $a(x, y) = 0$ if $|x_1| > \delta$, we only need to show that

$$\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^3} e^{i\lambda|x-y|} a(x_1, x', y) f(y) dy \right|^2 dx' \leq C |\lambda|^{-1} \|w\|_{\mathcal{F}(\mathbb{R}^3)} \int_{\mathbb{R}^3} |f|^2 w^{-1} dz \quad (5.8)$$

for any fixed $x_1 \in [-\delta, \delta]$. Here, $x' = (x_2, x_3) \in \mathbb{R}^2$.

To show (5.8), we first let

$$S_\lambda f(x') = \int_{\mathbb{R}^3} e^{i\lambda|x-y|} a(x_1, x', y) f(y) dy$$

for $x_1 \in [-\delta, \delta]$ fixed. Then, using the adjoint operator S_λ^* of S_λ , (5.8) follows easily from

$$\|w^{1/2} S_\lambda^* S_\lambda (w^{1/2} f)\|_{L^2(\mathbb{R}^3)} \leq C |\lambda|^{-1} \|w\|_{\mathcal{F}(\mathbb{R}^3)} \|f\|_{L^2(\mathbb{R}^3)}, \quad (5.9)$$

where

$$S_\lambda^* S_\lambda f(y) = \int_{\mathbb{R}^3} K_\lambda(y, z) f(z) dz$$

with

$$K_\lambda(y, z) = \int_{\mathbb{R}^2} e^{-i\lambda(|y-x|-|z-x|)} \overline{a}(x_1, x', y) a(x_1, x', y) dx'$$

such that

$$|K_\lambda(y, z)| \leq \frac{C}{1 + |\lambda||y - z|} \quad (5.10)$$

(see [32], p. 382). The key point here is that the kernel $K_\lambda(y, z)$ can be controlled by that of the fractional integral operator I_2 of order 2 given in (1.7). Indeed, by (5.10) it is clear that

$$|S_\lambda^* S_\lambda f(y)| \leq C |\lambda|^{-1} I_2(|f(z)|)(y).$$

Thus by the estimate (4.6) it follows that

$$\begin{aligned} \|S_\lambda^* S_\lambda f\|_{L^2(\mathbb{R}^3, w dy)} &\leq C |\lambda|^{-1} \|I_2(|f(z)|)\|_{L^2(\mathbb{R}^3, w dy)} \\ &\leq C |\lambda|^{-1} \|w\|_{\mathcal{F}(\mathbb{R}^3)} \|f\|_{L^2(\mathbb{R}^3, w^{-1} dz)} \end{aligned}$$

which is clearly equivalent to (5.9). The proof is now completed. \square

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